PMM U.S.S.R., Vol.49, No.3, pp. 290-295, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

APPLICATION OF THE METHOD OF TWO-SCALE EXPANSIONS TO THE SINGLE-FREQUENCY PROBLEM OF THE THEORY OF NON-LINEAR OSCILLATIONS *

V.D. AZHOTKIN and V.M. BABICH

The method of two-scale expansions is applied to a single-frequency system. The expansions obtained are justified over an asymptotically large time interval using the method of successive approximations.

1. Many authors have used the method of two-scale expansions and similar methods /1-5/ to construct solutions of the following system as $\varepsilon \to 0$:

$$\begin{aligned} d\varphi/dt &= \omega(I) + \varepsilon f(\varphi, I, \varepsilon), \quad \omega(I) > 0 \\ dI/dt &= \varepsilon g(\varphi, I, \varepsilon); \ \omega, f, g \in C^\infty, \ 0 < \varepsilon \leqslant 1 \end{aligned} \tag{1.1}$$

Below we propose a two-scale expansion, different from existing ones, of the solutions of system (1.1)

$$\begin{aligned} \varphi &= t_1 + \varepsilon\varphi_1(t_1, \tau) + \varepsilon^2\varphi_2(t_1, \tau) + \ldots \\ I &= I_0(\tau) + \varepsilon I_1(t_1, \tau) + \varepsilon^2 I_2(t_1, \tau) + \ldots \\ &\left(t_1 = \frac{\varphi_{-1}(\tau)}{\varepsilon} + \varphi_0(\tau), \quad \tau = \varepsilon t\right) \end{aligned} \tag{1.2}$$

where t_1 is the fast time, τ is the slow time, $\varphi_j (j \geqslant -1), I_j (j \geqslant 0)$ are the required functions, 2π periodic in time t_1 when $j \geqslant 1$. The proposed procedure is direct and does not require successive changes of variables, which simplifies the derivation of the asymptotic forms of the solution of system (1.1). Eqs (1.1), and also the equations for deriving I_0 (see below), are non-linear and their existence can only be guaranteed on a segment of the form $0 \leqslant \tau \leqslant \tau_0 < +\infty$, i.e., for $0 \leqslant t \leqslant \tau_0 \varepsilon$. On such segments, when ε is fairly small, it is possible to prove the existence of a true solution of the Cauchy problem for system (1.1), and expansions (1.2), in fact, provide the asymptotic forms of these solutions. Hence, whereever one succeeds in constructing series (1.2), a true solution of system (1.1) exists, and the asymptotic form of that solution is given by (1.2).

2. Let us construct series that asymptotically satisfy system (1.1) with initial conditions

$$\varphi|_{t=0} = a, \quad I|_{t=0} = b \tag{2.1}$$

We substitute (1.2) into (1.1) and equate terms of higher order on both sides of these equations

$$\varphi_{-1}'(\tau) = \omega(I_0), \quad I_1^{\boldsymbol{\cdot}}\varphi_{-1}' + I_0' = g(t_1, I_0, 0) \tag{2.2}$$

where the prime indicates a derivative with respect to τ, and a dot a derivative with respect to t_1. Averaging the second of Eq.(2.2) over the period 2π, we obtain

$$I_0'(\tau) = \langle g(t_1, I_0, 0)\rangle \quad \left(\langle F\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(t_1)\,dt_1\right) \tag{2.3}$$

Eq.(2.3) is the equation of the method of averaging /1,6/. From (2.3), the first of Eqs.(2.2), and initial conditions (2.1) we obtain

$$\varphi_{-1}(0) = 0, \quad \varphi_{-1}'(\tau) = \omega(I_0); \quad I_0(0) = b, \quad I_0'(\tau) = \langle g(t_1, I_0, 0)\rangle \tag{2.4}$$

from which in some segment $0 \leqslant \tau \leqslant \tau_0 < +\infty$ φ_{-1} and I_0 are uniquely defined.

Let us write down the terms of the following approximation:

$$\begin{aligned} &\varphi_0'(\tau) + \varphi_1^{\boldsymbol{\cdot}}\varphi_{-1}' = \omega(I_0) I_1 + f(t_1, I_0, 0); \quad \omega'(I_0) = \frac{d\omega(I_0)}{dI_0} \\ &I_2^{\boldsymbol{\cdot}}\varphi_{-1}' + I_1' = g^{\boldsymbol{\cdot}}(t_1, I_0, 0)\varphi_1 + \frac{\partial g(t_1, I_0, 0)}{\partial I_0} I_1 + \frac{\partial g(t_1, I_0, 0)}{\partial\varepsilon} \end{aligned} \tag{2.5}$$

*Prikl.Matem.Mekhan., 49, 3, 377-383, 1985

From (2.2) we obtain I_1, apart from the term that depends only on τ

$$I_1 = I_1^0(t_1, \tau) + I_1^1(\tau) \tag{2.6}$$

We select the first term I_1^0 so that $\langle I_1^0 \rangle = 0$. Eqs.(2.5) may be considered as equations for determining φ_1 and I_2 in the class of functions that are 2π periodic in t_1. The condition for such φ_1 and I_2 to exist is the equality of the left and right sides of (2.5) averaged over the period (in this case $\langle \varphi_1^{\cdot} \rangle = 0, \langle I_2^{\cdot} \rangle = 0$)

$$\varphi_0'(\tau) = \omega'(I_0) I_1^1(\tau) + \langle f \rangle, \quad I_1^1(\tau) = \langle g^{\cdot}\varphi_1 \rangle + \left\langle \frac{\partial g}{\partial I_0} I_1 \right\rangle + \left\langle \frac{\partial g}{\partial \varepsilon} \right\rangle \tag{2.7}$$

We eliminate the function φ_1 from the first term on the right side of the second of Eqs.(2.7) and obtain

$$\langle g^{\cdot}\varphi_1 \rangle = -\langle g\varphi_1^{\cdot} \rangle = -\left\langle \frac{g(t_1, I_0, 0)}{\omega(I_0)} (\omega'(I_0) I_1^1 + \right. \tag{2.8}$$
$$\left. \omega'(I_0) I_1^0 + f(t_1, I_0, 0) - \varphi_0'(\tau)) \right\rangle = [\langle g \rangle \langle f \rangle - \langle gf \rangle - \omega'(I_0) \langle g I_1^0 \rangle] / \omega(I_0)$$

In the transformations we used Eqs.(2.5) and (2.7). It follows from (2.6) and (2.8) that the second of Eqs.(2.7) is a first-order linear equation in $I_1^1(\tau)$. Its solution yields $I_1^1(\tau)$ and $\varphi_0(\tau)$. From (2.1) and (2.6) we obtain

$$\varphi_0(0) = a, \quad I_1^1(0) = -I_1^0(a, 0) \tag{2.9}$$

The initial conditions (2.9) and Eqs.(2.7) and (2.8) define φ_0 and I_1^1.

3. Suppose $\varphi_{-1}, \varphi_0, \ldots, \varphi_{j-1}, I_0, I_1, \ldots, I_j$ have been found and the equation $d\varphi/dt = \omega(I) + \varepsilon f$ (respectively $dI/dt = \varepsilon g$ see (1.2)) has been satisfied with an accuracy to terms of order ε^{j-1}. The initial data of the Cauchy problem (2.1) are satisfied for φ, I with an accuracy to terms of order ε^{j-1}. We shall now deal with terms of the following approximation:

$$\varphi_j^{\cdot}\omega(I_0) = A_j(t_1, \tau), \quad I_{j+1}^{\cdot}\omega(I_0) = g^{\cdot}\varphi_j + B_j(t_1, \tau) \tag{3.1}$$

where A_j and B_j are 2π periodic functions of t_1, dependent on $\varphi_{i-1}, I_i, i \leqslant j$. For the periodic solution φ_j, I_{j+1} of system (3.1) to exist it is necessary and sufficient that the equations

$$\langle A_j \rangle = 0, \quad \langle -gA_j / \omega(I_0) + B_j \rangle = 0 \tag{3.2}$$

are satisfied.

We separate in the functions φ_j, I_{j+1} the "fast" and "slow" parts

$$\varphi_j = \varphi_j^0(t_1, \tau) + \varphi_j^1(\tau); \quad \langle \varphi_j^0 \rangle = 0 \tag{3.3}$$
$$I_{j+1} = I_{j+1}^0(t_1, \tau) + I_{j+1}^1(\tau); \quad \langle I_{j+1}^0 \rangle = 0$$

which depend only on τ.

We integrate (3.1) with respect to t_1, and taking into account (3.3) obtain

$$\varphi_j^0(t_1, \tau) = c_j - \langle c_j \rangle, \quad c_j = \int_0^{t_1} \frac{A_j}{\omega(I_0)} dt_1 \tag{3.4}$$

$$I_{j+1}^0(t_1, \tau) = \frac{g - \langle g \rangle}{\omega(I_0)} \varphi_j^1 - D_j - \langle D_j \rangle, \quad D_j = \int_0^{t_1} \frac{B_j + g^{\cdot}\varphi_j^0}{\omega(I_0)} dt_1$$

The functions $\varphi_j^1(\tau)$ and $I_{j+1}^1(\tau)$ will be defined using equations of subsequent approximation and initial data. Let us write these equations

$$\varphi_{j+1}^{\cdot}\omega(I_0) - \varphi_j' + \varphi_j^{\cdot}\varphi_0' = \omega(I_0) I_{j+1} + f^{\cdot}\varphi_j^1 + E_{j+1} \tag{3.5}$$
$$I_{j+1}^{\cdot}\omega(I_0) + I_j^{\cdot}\varphi_0' - I_j' = g^{\cdot}\varphi_{j+1} + \frac{\partial g}{\partial I_0} I_{j+1} +$$
$$\lambda_j g^{\cdot\cdot}\varphi_1\varphi_j + \frac{\partial g^{\cdot}}{\partial I_0} I_1\varphi_j^1 + F_{j+1}; \quad \lambda_1 = \frac{1}{2}; \quad \lambda_j = 1 \quad (j > 1)$$

where E_{j+1}, F_{j+1} are known functions of t_1, τ, periodic in t_1. Averaging Eqs.(3.5) over t_1, we obtain the required equations

$$\varphi_j^{1\prime} = \omega'(I_0) I_{j+1}^1(\tau) + \langle E_{j+1} \rangle \tag{3.6}$$
$$I_{j+1}' = \left\langle g^{\cdot\cdot}\varphi_1^0 + \frac{\partial g^{\cdot}}{\partial I_0} I_1^0 \right\rangle \varphi_j^1 + \left\langle \frac{\partial g}{\partial I_0} \right\rangle I_{j+1}^1 + \langle c_{j+1} \rangle$$

where c_{j+1} is a known function. The coefficient of φ_j^1 is transformed to

$$-\frac{1}{\omega(I_0)}\langle g^{\cdot}f\rangle+\frac{\omega(I_0)}{2}\frac{\partial}{\partial I_0}\left(\frac{\langle g\rangle^2-\langle g^2\rangle}{\omega^2(I_0)}\right)$$

We shall now require the initial data to be satisfied for $\varphi(1)$ with an accuracy to terms of order ε^j (ε^{j+1})

$$\varphi_j^1(0)+\varphi_j^0(a,0)=0,\quad I_{j+1}^1(0)+I_{j+1}^0(a,0)=0 \tag{3.7}$$

Formulae (3.7) determine the initial data for system (3.6). The quantities φ_j^1 and I_{j+1}^1 are uniquely defined by (3.6). Eqs.(3.5) now take the form (3.1) with j replaced by $j+1$, and conditions (3.2) (with j replaced by $j+1$) also hold. The process of constructing I_{j+1}, φ_j can be continued.

Note that the slow variable I is determined at each step with an accuracy one order of ε greater than the fast variable φ. This situation is characteristic when using the method of perturbations in the theory of oscillations.

4. Now, since the two-scale expansions of the Cauchy problem (1.1), (2.1) have been constructed, the question of their substantiation arises. It consists of proving the solvability of problem (1.1), (2.1) in an asymptotically large time interval, and evaluating the remainder terms. To obtain these results the classical method of successive approximations is convenient (see /7/). In particular, in /7/, the method of averaging for Eq.(1.1) is proved (differently from here). Other methods and their proof can be found in /1,6/.

We shall now restate the results obtained. Let us assume that f, g are 2π periodic functions of φ and

$$f,\ g,\ \omega\in C^\infty(-\infty<\varphi<+\infty,\ |I-b|\leqslant\alpha,\ 0\leqslant\varepsilon\leqslant\beta,$$
$$I|_{t=0}=b,\ 0<\alpha=\text{const},\ 0<\beta=\text{const})$$

Let problem (2.3), (2.4) be solvable for $I_0(\tau)$ when $0\leqslant\tau\leqslant\tau_0<+\infty$, and $|I_0(\tau)-b|<\alpha$, $0\leqslant\tau\leqslant\tau_0$. When $0\leqslant\tau\leqslant\tau_0$ all terms of series (1.2) can be constructed under these conditions.

Theorem. There exists an ε_0, $0<\varepsilon_0\leqslant\beta$, such that when $0\leqslant t\leqslant\tau_0/\varepsilon$, problem (1.1), (2.1) is solvable, provided $|I(t)-b|<\alpha$.

Series (1.2) obtained above give the asymptotic form of the solution of the Cauchy problem in the following sense. Let φ and I be the solution of the Cauchy problem (1.1), (2.1). We introduce the terms R_φ and R_I using the equations

$$\varphi=t_1+\sum_{j=1}^{r}\varepsilon^j\varphi_j+R_\varphi,\quad I=\sum_{j=0}^{r}\varepsilon^jI_j+R_I \tag{4.1}$$

The following inequalities

$$|R_\varphi|\leqslant\text{const}\,\varepsilon^{r-1};\ |R_I|\leqslant\text{const}\,\varepsilon^{r+1},\ 0\leqslant\varepsilon\leqslant\varepsilon_0 \tag{4.2}$$

are satisfied for $0\leqslant t\leqslant\tau_0\,\varepsilon$.

It turns out that the investigation is simplified if instead of R_φ and R_I the remainder terms S_φ and S_I are introduced so that

$$\varphi=t_1+\sum_{j=1}^{r+2}\varepsilon^j\varphi_j+S_\varphi,\quad I=\sum_{j=0}^{r+2}\varepsilon^jI_j+S_I \tag{4.3}$$

derive for S_I and S_φ a system of integral equations, and prove in the interval $0\leqslant t\leqslant\tau_0/\varepsilon$ the comparatively weak estimate

$$|S_\varphi|\leqslant\text{const}\,\varepsilon^{r+1},\ |S_I|\leqslant\text{const}\,\varepsilon^{r+1} \tag{4.4}$$

By virtue of the obvious equations

$$\varepsilon^{r+1}\varphi_{r+1}+\varepsilon^{r+2}\varphi_{r+2}+S_\varphi=R_\varphi,\quad \varepsilon^{r+1}I_{r+1}+\varepsilon^{r+2}I_{r+2}+S_I=R_I \tag{4.5}$$

the estimates (4.4) are sufficient for obtaining the estimates (4.2). From (1.1) and (4.3) there follows the equation for S_φ and S_I

$$\frac{dS_\varphi}{dt}=-\frac{d}{dt}(t_1+\ldots+\varepsilon^{r+2}\varphi_{r+2})+\omega(I_0+\ldots+S_I)+ \tag{4.6}$$
$$\varepsilon f(t_1+\ldots+S_\varphi,I_0+\ldots+S_I,\varepsilon)$$
$$\frac{dS_I}{dt}=-\frac{d}{dt}(I_0+\ldots+\varepsilon^{r+2}\varphi_{r+2})+\varepsilon g(t_1+\ldots+S_\varphi,I_0+\ldots+S_I,\varepsilon)$$

The initial data for S_φ and S_I are zero

$$S_\varphi|_{t=0}=0, \quad S_I|_{t=0}=0 \tag{4.7}$$

since the sums

$$t_1+\sum_{j=1}^{r+2}\varepsilon^j\varphi_j, \quad \sum_{j=0}^{r+2}\varepsilon^j I_j$$

satisfy the initial data (2.1). On the right sides of (4.6) we separate the free terms Φ_0, Ψ_0, the terms Φ_1, Ψ_1 linear relative to S_φ and S_I (in expansion of the right sides in Maclaurin series in powers of S_φ and S_I), and the remainder terms Φ_2, Ψ_2 which have a quadratic estimate for small S_φ and S_I. Taking into account the initial data (4.6) for S_φ and S_I, we replace (4.7) by the integral equations

$$S_\varphi=\int(\Phi_0+\Phi_1+\Phi_2)\,dt', \quad S_I=\int(\Psi_0+\Psi_1+\Psi_2)\,dt' \tag{4.8}$$

Here and henceforth integration with respect to t' is carried out from $t'=0$ to $t'=t$. On the assumption that

$$0\leqslant t\leqslant \tau_0/\varepsilon, \quad |S_\varphi|\leqslant A_0, \quad |S_I|\leqslant A_0, \quad 0\leqslant\varepsilon\leqslant\beta_1\leqslant\beta \tag{4.9}$$

(the numbers A_0 and β_1 must be so small that the inequality $|I-b|<\alpha$ is not violated), we have

$$|\Phi_0|\leqslant A_1\varepsilon^{r+3}, \quad |\Phi_2|\leqslant A_1(S_I^2+S_\varphi^2) \tag{4.10}$$

$$|\Psi_0|\leqslant A_1\varepsilon^{r+3}, \quad |\Psi_2|\leqslant A_1\varepsilon(S_I^2+S_\varphi^2), \quad A_1=\text{const}>0$$

$$\Phi_1=\omega' S_I+\varepsilon\frac{\partial f}{\partial\varphi}S_\varphi+\varepsilon\frac{\partial f}{\partial I}S_I, \quad \Psi_1=\varepsilon\left(\frac{\partial g}{\partial\varphi}S_\varphi+\frac{\partial g}{\partial I}S_I\right)$$

The arguments of the functions ω', f_φ', f_I', g_φ', g_I' are

$$t_1+\ldots+\varepsilon^{r+2}\varphi_{r+2}, \quad I_0+\ldots+\varepsilon^{r+2}I_{r+2}, \varepsilon$$

We apply the method of successive approximations not to system (4.8), but to some identical transformation of it. The terms $\int\Phi_1 dt'$ and $\int\Psi_1 dt'$ are insufficiently "weak" for a simple substantiation of the method of successive approximations. Let us try to find a transformation of (4.8) which would eliminate the linear terms from it. We begin the transformation from the second equation of (4.8). The principal term (in the group of linear terms) which contains S_φ is

$$\int\frac{\partial g(t_1,I_0,0)}{\partial\varphi}S_\varphi dt'=\int\frac{\partial g(t_1,I_0,0)}{\partial t'}\left(\frac{dt_1}{dt}\right)^{-1}S_\varphi dt'$$

We integrate this integral by parts, eliminating the function g from the derivative with respect to t'. We change in the terms linear in S_φ on the right side of the second Eq.(4.8), the function S_φ by its expression from the first of Eqs.(4.8).

We convert the new equation which has S_I as its left side, as follows: assuming on the right side that all terms, apart from the linear uniform ones containing S_I are known, we solve this equation for S_I. We arrive at an equation for S_I not containing on the right side linear Volterra operators in S_I. We substitute the expression obtained for S_I in the appropriate place into the terms linear in S_I of the first of Eqs.(4.8). Now the linear terms of this equation will not contain S_I. We arrive at a system of equations for S_φ and S_I, the linear uniform terms of which (in the right sides of the equations) are Volterra operators that have the form

$$KF=\varepsilon\int K(t,t',\varepsilon)F(t')\,dt', \quad 0\leqslant t'\leqslant t\leqslant\frac{\tau_0}{\varepsilon}$$

whose kernels $K(t, t', \varepsilon)$ are bounded.

Using successive approximations to solve this system, assuming all terms on the right, apart from those linear and uniform in S_φ and S_I, are known, we obtain a system of integral equations of the form

$$S_\varphi=\int(\Phi_4+\Phi_5)\,dt', \quad S_I=\int(\Psi_4+\Psi_5)\,dt', \quad 0\leqslant t\leqslant\frac{\tau_0}{\varepsilon} \tag{4.11}$$

where Φ_4 and Ψ_4 are independent of S_φ and S_I, and when inequalities (4.9) are satsified, the following estimates hold:

$$|\Phi_4|\leqslant A_2\varepsilon^{r+3}, \quad |\Psi_4|\leqslant A_2\varepsilon^{r+3} \tag{4.12}$$

$$|\Phi_5|\leqslant A_2(S_I^2+\varepsilon S_\varphi^2), \quad |\Psi_5|\leqslant A_2\varepsilon(S_I^2+S_\varphi^2)$$

When carrying out the transformations it is necessary to bear in mind that the composition of integral operators of the form

$$K_jF = \sqrt{\varepsilon}\int K_j(t, t', \varepsilon)\, F(t')\, dt', \quad 0 \leqslant t \leqslant \frac{\tau_0}{\varepsilon}, \quad j = 1, 2$$

with uniformly bounded kernels K_j gives the integral operator K_1K_2, whose kernel is uniformly bounded when $0 \leqslant t' \leqslant t \leqslant \tau_0/\varepsilon$.

5. We will solve the system of integral Eqs.(4.12) by the usual method of successive approximations, setting

$$\begin{aligned} S_q^{n+1} &= \int (\Phi_4 + \Phi_5 (S_q^n, S_I^n, t', \varepsilon))\, dt', \quad S_q^{-1} = 0 \\ S_I^{n+1} &= \int (\Psi_4 + \Psi_5 (S_q^n, S^n_I, t', \varepsilon))\, dt', \quad S_I^{-1} = 0 \end{aligned} \tag{5.1}$$

It can be shown that when $0 < \varepsilon \leqslant \varepsilon_0$, where ε_0 is a fairly small number, the successive approximations will not go beyond the limits of the square

$$| S_q | \leqslant A_3 \varepsilon^{r+1}, \quad | S_I | \leqslant A_3 \varepsilon^{r+1} \tag{5.2}$$

where A_3 is an arbitrary but fixed number.

To prove estimate (5.2), we use the inequalities

$$\begin{aligned} | S_q^{n+1} | &\leqslant A_2 \int (\varepsilon^{r+3} - (S_I^n)^2 + \varepsilon (S_q^n))\, dt' \\ | S_I^{n+1} | &\leqslant A_2 \int (\varepsilon^{r+3} + \varepsilon (S_I^n)^2 - \varepsilon (S_q^n)^2)\, dt' \end{aligned} \tag{5.3}$$

which hold when inequalities (4.9) are satisfied and follow from estimates (4.12).

The next step for obtaining estimates (5.2) is based on the following consideration (a similar method was used previously /8/). Let L and M be non-negative solutions of the system of integral equations

$$L = A_2 \int (\varepsilon^{r+3} - M^2 + \varepsilon L^2)\, dt', \quad M = A_2 \int (\varepsilon^{r+3} - \varepsilon M^2 + \varepsilon L^2)\, dt' \tag{5.4}$$

If for some n we have $| S_q^n | \leqslant L$, $| S_I^n | \leqslant M$, then, as follows from (5.3) and (5.4), we have for all $n' > n$ the inequalities $| S_q^{n'} | \leqslant L$, $| S_I^{n'} | \leqslant M$. Hence it is sufficient to prove the existence of the solution L, M of system (5.4) such that

$$0 \leqslant L \leqslant A_3\varepsilon^{r+1}, \quad 0 \leqslant M \leqslant A_3\varepsilon^{r+1} \tag{5.5}$$

Let us prove that such L and M exist. We make in (5.4) the substitution $L = \varepsilon^{r+1} l$, $M = \varepsilon^{r+1} m$. The square (5.2) becomes $| l | \leqslant A_3$, $| m | \leqslant A_3$.

The equations for l and m are obviously of the form

$$\begin{aligned} l &= A_2 (\varepsilon^2 t - \varepsilon^{r-1} \int (m^2 - \varepsilon l^2)\, dt') \\ m &= A_2 (\varepsilon^2 t + \varepsilon^{r+2} \int (m^2 + l^2)\, dt') \end{aligned}$$

This system of integral equation is equivalent ot the system of differential equations with zero initial conditions

$$\begin{aligned} &\frac{dl}{d\tau} = A_2(\varepsilon + \varepsilon^2 (m^2 + \varepsilon l^2)), \quad \frac{dm}{d\tau} = A_2 (\varepsilon + \varepsilon^{r-1} (m^2 + l^2)) \\ &\tau = \varepsilon t; \quad l(0) = m(0) = 0 \end{aligned} \tag{5.6}$$

We will consider these equations for $0 \leqslant \tau \leqslant \tau_0$ and $| l | \leqslant A_3$, $| m | \leqslant A_3$, assuming that $r > 0$. The last requirement does not limit the generality. The existence of a non-negative solution of this system follows from Picard's theorem, when $| l | \leqslant A_3$, $| m | \leqslant A_3$ with zero initial conditions when $0 \leqslant \tau \leqslant \min \{\tau_0, A_3 N\}$, where N is the maximum of the modulus of the right side when $| l | \leqslant A_3$, $| m | \leqslant A_3$. Since $0 \leqslant N \leqslant \mathrm{const} + \varepsilon$, then for fairly small ε_0 $(0 < \varepsilon \leqslant \varepsilon_0)$ the quantity $A_3 N$ does not exceed τ_0 and a solution exists when $0 \leqslant \tau \leqslant \tau_0$.

The existence of non-negative l and m in the square $| l | \leqslant A_3$, $| m | \leqslant A_3$ (and consequently $L = \varepsilon^{r-1} l$, $M = \varepsilon^{r-1} m$ in the square (5.5)) is thus proved. At the same time the inequalities

$$| S_q^n | \leqslant L \leqslant A_3\varepsilon^{r-1}, \quad | S_I^n | \leqslant M \leqslant A_3\varepsilon^{r-1} \tag{5.7}$$

are proved for any n, since they obviously are satisfied when $n = -1$.

The proof of the uniform convergence of the successive approximations S_q^n and S_I^n to limits as $n \to \infty$, after the proof of inequality (5.7) is not difficult, and is carried out as in Picard's theorem.

Passing to the limit in (5.7) as $n \to \infty$, we obtain the estimates (4.4) with const $= A_3$, which completes the proof of the method of two-scale expansions.

The extension of the results of this paper to the case when $I = (I^1, I^2, \ldots, I^s)$, $s > 1$, is trivial. For the multifrequency case $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^s)$, $s > 1$ there is no such simple and complete theory as in the case of $s = 1$.

The authors thank A.M.Il'in for pointing out the method of eliminating the remainder terms when proving the theorem.

REFERENCES

1. BOGOLYUBOV N.N. and MITROPOL'SKII YU.A., Asymptotic Methods in the Theory of Non-Linear Oscillations. Moscow, Nauka, 1974.
2. NAIFEH A.H., Perturbation Methods. Moscow, MIR, 1976.
3. DOBROKHOTOV S. YU. and MASLOV V.P., Finite-zone almost periodic solutions in the Wentzel-Kramers-Brillouin approximation. In: Itogi Nauki i Tekhniki. Ser. Sovremennye Problemy Matematiki, Vol.15, Moscow, VINITI, 1980.
4. KUZMAN G.E., Asymptotic solutions of non-linear second-order differential equations with variable coefficients. PMM, Vol.23, No.2, 1959.
5. COULL J.D., The Method of Perturbations in Applied Mechanics. Moscow, Mir, 1972.
6. ARNOL'D V.I., Supplementary Chapters of the Theory of Ordinary Differential Equations. Moscow, Nauka, 1978.
7. AKULENKO L.D., Application of the method of averaging and successive approximations to the, investigation of non-linear oscillations. PMM, Vol.45, No.5, 1981.
8. SOBOLEV S.L., Some Applications of Functional Analysis in Mathematical Physics. Leningrad, Izd. LGU, 1950.

Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.2,pp.295-301,1985 0021-8928/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

ON THE CONDITIONS FOR THE EXISTENCE OF THE REDUCING CHAPLYGIN FACTOR*

IL. ILIYEV

The problem of the existence of a reducing Chaplygin factor (RCF) for non-holonomic systems with k degrees of freedom is discussed. By introducing additional coordinates, a class of non-holonomic systems for which the RCF method is applicable in a widened configuration space is distinguished. For comparison, the corresponding conditions in quasi-coordinates are given. The existence of an RCF for one of the equivalent non-holonomic systems is studied.

1. Formulation of the problem. S.A. Chaplygin formulated the conditions under which non-holonomic systems with two degrees of fredom can have a reducing factor (see /1/). Using the equations in admissible vectors, Chaplygin's ideas were extended to systems which have k degrees of freedom, /2/. The present paper continues the investigations initiated in /2/.

Let us recall from /2/ some of the equations necessary for our discussion. We assume for brevity that the indices $\lambda, \mu, \nu, \varkappa, \rho, \ldots$ take values from 1 to n; a, b, c, d from 1 to k; and $p, q, r, \ldots$ from k to n.

By means of

$$d\tau = N(q^\alpha)\, dt \tag{1.1}$$

the equations of motion of a non-holonomic system in admissible vectors,

$$\frac{d}{dt}\left(\frac{\partial \theta}{\partial s^{\cdot a}}\right) - \frac{\partial \theta}{\partial q^{\varkappa}} \alpha_a{}^{\varkappa} - \Delta_{a,\, bc} s^{\cdot b} s^{\cdot c} = \frac{\partial L}{\partial q^{\varkappa}} \alpha_a{}^{\varkappa} \tag{1.2}$$

is changed to the form

$$\frac{d}{d\tau}\left(\frac{\partial (\theta)}{\partial s^{\prime a}}\right) - \frac{\partial (\theta)}{\partial q^{\varkappa}} \alpha_a{}^{\varkappa} = \frac{\partial L}{\partial q^{\varkappa}} \alpha_a{}^{\varkappa} \tag{1.3}$$

Prikl. Matem. Mekhan.,49,3,384-391,1985